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On Phillips Operator

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1. INTRODUCTION

The Phillips operator is defined by

$$S_\lambda(f, t) = \int_0^\infty e^{-\lambda(t+u)} \left(\sum_{n=1}^\infty \frac{(\lambda^2 t)^n u^{n-1}}{n!\,(n-1)!} \right) f(u)\, du + e^{-\lambda t} f(0). \qquad (1.1)$$

As far as the degree of approximation is concerned, the behavior of this operator is very similar to that of Bernstein polynomials, Szasz operators, and Post–Widder operators, or, following [7], the so-called exponential-type operators. However, the Phillips operator does not satisfy the differential equation

$$(\partial/\partial t)\, W(\lambda, t, u) = (\lambda/p(t))\, W(\lambda, t, u)(u - t), \qquad (1.2)$$

where $W(\lambda, t, u)$ is the kernel of S_λ (that is

$$W(\lambda, t, u) = e^{-\lambda(t+u)} \left(\sum_{n=1}^\infty \frac{(\lambda^2 t)^n u^{n-1}}{n!\,(n-1)!} + \delta(u) \right) \qquad (1.3)$$

and

$$S_\lambda(f, t) = \int_0^\infty W(\lambda, t, u) f(u)\, du). \qquad (1.4)$$

Since relation (1.2) is one of the basic properties of the exponential operators, there are some technical difficulties in the analogous estimations for the Phillips operators. In this note, we try to suggest some methods for handling such problems by discussing the properties of the Phillips operators. This is our main purpose.

In the study of the approximation properties of the Phillips operators, a few results for some special cases are known. In the case when f is replaced by

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$T(\cdot)f$ in (1.1), where $\{T(t), t \geq 0\}$ is a C_0-semigroup of operators. Phillips [8] proved that $S_\lambda(T(\cdot)f, t)$ converges to $T(t)f$ uniformly on $[0, L]$ for any $0 < L < \infty$. Recently, Ditzian [3] estimated this rate of convergence in terms of $\omega_L(\lambda^{-(1/2)}, T(\cdot)f)$, the modulus of continuity of $T(\cdot)f$ on $[0, L]$. Later, Ditzian and May [4] characterized the saturation class for this operator, again when f is of the form $T(\cdot)f$. In the present note, we determine the classes of functions satisfying $\|S_\lambda(f, k, t) - f(t)\|_{C[a,b]} = O(\lambda^{-\alpha(k+1)/2})$, for all $0 < \alpha \leq 2$, where $S_\lambda(f, k, t)$ is a combination of the Phillips operators $S_\lambda(f, t)$, and f is a function in $C[0, \infty)$. In particular when $k = 0$, we obtain both saturation and inverse theorems for Phillips operators. The analogous results for semigroups of operators can be deduced from it easily.

The saturation theorem (i.e., $\alpha = 2$) proved in Section 2 is essentially part of the author's Ph.D. thesis which was written under the supervision of Professors Z. Ditzian and S. Riemenschneider.[1] In Section 3, a direct theorem is proved for the purpose of the completeness of the theory. In Section 4, we prove an inverse theorem, which the author was not able to prove in his thesis. For simplicity, we only prove the case when $k = 0$ for the inverse theorem, but there is no essential difficulties for the generalization.

2. The Saturation Result

We first prove some preliminary results.

Let $W(\lambda, t, u)$, defined by (1.3), be the kernel of the Phillips operator. The following lemma, describing the "varying property" of the kernel, is the basic property of this operator.

LEMMA 2.1. *If*

$$\mathbb{P} = t\left(1 - \frac{D}{\lambda}\right)^2 = t\left(1 - \frac{2}{\lambda}D + \frac{1}{\lambda^2}D^2\right), \tag{2.1}$$

where $D = \partial/\partial t$, *then*

$$\mathbb{P}W(\lambda, t, u) = W(\lambda, t, u)\,u. \tag{2.2}$$

Proof. Beginning with the expression

$$e^{\lambda t}S_\lambda(f, t) = \int_0^\infty e^{-\lambda u}\left[\sum_{n=1}^{\infty} \frac{(\lambda^2 t)^n u^{n-1}}{n!\,(n-1)!} + \delta(u)\right] f(u)\,du,$$

[1] Professor Ditzian was the author's thesis supervisor and Professor Riemenschneider was the author's direct supervisor in the year when Professor Ditzian was on sabbatical.

we differentiate twice with respect to t on both sides to obtain

$$e^{\lambda t}(\lambda + D)^2 S_\lambda(f, t) = (\lambda^2/t) \int_0^\infty e^{\lambda t} W(\lambda, t, u)\, uf(u)\, du.$$

Dividing both sides by $(\lambda^2/t)\, e^{\lambda t}$ yields the required relation.

Remark. In order to calculate $S_\lambda(x^i, t)$, it would be simpler to compute $\mathbb{P}^i 1$ than to calculate directly from the operator. For example, $\mathbb{P}^0 1 = 1$, $\mathbb{P}^1 1 = t$, $\mathbb{P}^2 1 = t^2 + (2/\lambda)\, t$, $\mathbb{P}^{10} 1 = t^{10} + 90(t^9/\lambda) + 3240(t^8/\lambda^2) + 60480(t^7/\lambda^3) + 635040(t^6/\lambda^4) + 3810240(t^5/\lambda^5) + 12700800(t^4/\lambda^6) + 21772800(t^3/\lambda^7) + 16329600(t^2/\lambda^8) + 3628800(t/\lambda^9)$.

LEMMA 2.2. *Let $A_m(\lambda, t)$ be defined by*

$$A_m(\lambda, t) = \lambda^m \int_0^\infty W(\lambda, t, u)(u - t)^m\, du, \qquad m = 0, 1, 2, \ldots \tag{2.3}$$

Then the following statements are true.

(1) *$A_m(\lambda, t)$ is a polynomial in (λt);*

(2) *the degrees of $A_m(\lambda, t)$ in (λt) is $[m/2]$, while the degree of $S_\lambda(x^m, t)$ in t is exactly m.*

Proof. First observe that, if $f, g \in C^\infty$, then

$$\mathbb{P}(f \cdot g) = (\mathbb{P}f) \cdot g + \frac{2}{\lambda} tf \cdot (D \cdot g) - \frac{t}{\lambda^2} f \cdot (D^2 g) + \frac{t}{\lambda^2} D[f \cdot Dg]. \tag{2.4}$$

This can be seen from the relation

$$\mathbb{P}(f \cdot g) = t\left(1 + \frac{2}{\lambda} D + \frac{D^2}{\lambda^2}\right) f \cdot g = t\left\{fg + \frac{2}{\lambda}(Df)\, g + \frac{1}{\lambda^2}(D^2 f)\, g + \frac{2}{\lambda} f\, Dg + \frac{1}{\lambda^2} f\, D^2 g + \frac{2}{\lambda^2}(Df)(Dg)\right\}.$$

Replacing $(Df)(Dg)$ by $D(f\, Dg) - f(D^2 g)$, we obtain relation (2.4).

Now the lemma can be proved easily by induction.

The lemma is trivial for $m = 0$, 1, and 2. Assume that the lemma is true for integers less than or equal to m. Applying the operator $((2t/\lambda)\, D + (t/\lambda^2)\, D^2) = \mathbb{P} - t$ to both sides of (2.3), we have

$$\begin{aligned}&\frac{2t}{\lambda} DA_m(\lambda, t) + \frac{t}{\lambda^2} D^2 A_m(\lambda, t)\\ &\qquad = \mathbb{P}\lambda^m \int_0^\infty W(\lambda, t, u)(u - t)^m\, du - t\lambda^m \int_0^\infty W(\lambda, t, u)(u - t)^m\, du\end{aligned}$$

$$= \lambda^m \left\{ \int_0^\infty [\mathbb{P} W(\lambda, t, u)](u-t)^m \, du - t\lambda^m \int_0^\infty W(\lambda, t, u)(u-t)^m \, dt \right\}$$

$$= 2t\lambda^{m-1} \int_0^\infty W(\lambda, t, u) \, D(u-t)^m \, du$$

$$+ t\lambda^{m-1} \int_0^\infty W(\lambda, t, u) \, D^2(u-t)^m \, du$$

$$+ 2t\lambda^{m-2} \int_0^\infty W(\lambda, t, u) \, D(u-t)^m \, du$$

$$= \frac{1}{\lambda} A_{m+1}(\lambda, t) - 2tmA_{m-1}(\lambda, t) - tm(m-1) A_{m-2}(\lambda, t)$$

$$+ \frac{2tm}{\lambda} DA_{m-1}(\lambda, t).$$

Thus, the lemma is also true for $m + 1$.

COROLLARY. *Let $W(\lambda, t, u)$, defined in (1.3), be the kernel of the Phillips operators. Let also N and δ be two positive numbers, and $[a, b]$ be any bounded interval. Then for any $m > 0$, there exists a constant M_m, such that*

$$\left\| \int_{|u-t| \geq \delta} W(\lambda, t, u) \, e^{Nu} \, du \right\|_{C[a,b]} \leq M_m \lambda^{-m}. \tag{2.5}$$

Our saturation result is for a linear combination of S_λ introduced by Butzer [2]. As a special case, when $k = 0$, the combination $S_\lambda(f, 0, t)$ reduces to the operator $S_\lambda(f, t)$, and we have a saturation theorem for the Phillips operators. The linear combination is defined as follows.

DEFINITION 2.3. The linear combination $S_\lambda(f, k, t)$ is defined by

$$S_\lambda(f, k, t) = \sum_{j=0}^{k} C(j, k) \, S_{2^j\lambda}(f, t), \tag{2.6}$$

where

$$C(j, k) = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{2^j}{2^j - 2^i}.$$

DEFINITION 2.4. Let $C_N[0, \infty) = \{f \in C[0, \infty) : |f(t)| \leq Me^{Nt}$ for some $M\}$. Define $\|\cdot\|_{C_N}$ on $C_N[0, \infty)$ by

$$\|f\|_{C_N} = \sup_{0 \leq t < \infty} |f(t)| \, e^{-Nt}.$$

It is easy to see that the operator $S_\lambda(f, k, t)$ is indeed an approximation operator for functions in $C_N[0, \infty)$ for some $N > 0$. It is also easy to see that, the rates of convergence to zero for $S_\lambda(f, k, t) - f(t)$ and for $S_{2\lambda}(f, k, t) - S_\lambda(f, k, t)$ are the same. However, dealing with $\{S_{2\lambda}(f, k, t) - S_\lambda(f, k, t)\}$ instead of $\{S_\lambda(f, k, t) - f(t)\}$ will simplify much of the proof of Lemma 2.8, which is one of the major steps for proving the saturation theorem.

LEMMA 2.5. *Let $f \in C_N[0, \infty)$ for some $N > 0$. If $f^{(2k+2)}(t)$ exists, then*

$$\lambda^{k+1}(S_{2\lambda}(f, k, t) - S_\lambda(f, k, t)) = \sum_{j=k+1}^{2k+2} Q(j, k, t) f^{(j)}(t) + o(1)$$
$$=: P_{2k+2}(D) f(t) + o(1), \tag{2.7}$$

where $\{Q(j, k, t)\}$ are polynomials in t. Moreover,

$$Q(2k+2, k, t) = C_1 t^{k+1} \quad \text{and} \quad Q(2k+1, k, t) = C_2 t^k.$$

If $f \in C^{2k+2}[a, b]$, then (2.7) is uniform in any interial interval $[a_1, b_1] \subset (a, b)$.

Proof. Clearly, $S_\lambda(e^{Nx}, t)$ is uniformly bounded for t in any bounded interval. Hence, by the corollary to Lemma 2.2, we may assume that f has compact support.

As $S_\lambda(f, k, t)$ is a linear combination of $S_\lambda(f, t)$, so is $S_{2\lambda}(f, k, t) - S_\lambda(f, k, t)$. If we denote

$$S_{2\lambda}(f, k, t) - S_\lambda(f, k, t) = \sum_{j=0}^{k+1} \alpha(j, k)\, S_{2^j\lambda}(f, t),$$

then the coefficients $\alpha(j, k)$ have the following property

$$\sum_{j=0}^{k+1} \alpha(j, k)\, 2^{-mj} = 0, \qquad \text{for} \quad m = 0, 1, \ldots, k. \tag{2.8}$$

This follows from the well-known fact that

$$\sum_{j=0}^{k} C(j, k)\, 2^{-mj} = 1 \qquad m = 0,$$
$$= 0 \qquad m = 1, 2, \ldots, k.$$

Now by Taylor's formula we write

$$f(u) = \sum_{j=0}^{2k+2} \frac{f^{(j)}(t)}{j!}(u - t)^j + \epsilon(u, t)(u - t)^{2k+2}.$$

Using Lemma 2.2 and relation (2.8), the lemma can be proved easily.

The following lemma is an induction step for proving the saturation theorem. This Lemma is similar to a lemma we proved in [5].

LEMMA 2.6. *Let $f \in C_N[0, \infty)$ for some $N > 0$. If $\lambda^{k+1} \| S_{2\lambda}(f, k, \cdot) - S_\lambda(f, k, \cdot)\|_{C[a,b]} \leqslant M_1$, then $\lambda^k \| S_{2\lambda}(f, k-1, \cdot) - S_\lambda(f, k-1, \cdot)\|_{C[a,b]} \leqslant M_2$.*

Proof. By Definition 2.3, it is easy to verify the following recursion relation

$$S_\lambda(f, k, t) = (2^k - 1)^{-1} [2^k S_{2\lambda}(f, k-1, t) - S_\lambda(f, k-1, t)]. \tag{2.9}$$

Let $\lambda = 2^n \lambda_0$, $\lambda_0 \in [N + 1, N + 2)$ and consider

$$I_n(t) = (2^k - 1) \sum_{i=0}^{n-1} 2^{ki} [S_{2^{i+1}\lambda_0}(f, k, t) - S_{2^i\lambda_0}(f, k, t)] \tag{2.10}$$

By assumption of the lemma, we have

$$\lambda_0^k \| I_n(t) \|_{C[a,b]} \leqslant (2^k - 1) \sum_{i=0}^{n-1} 2^{ki} (2^i \lambda_0)^{-(k+1)} M_1 \lambda_0^k \leqslant 2(2^k - 1) M_1. \tag{2.11}$$

On the other hand, substitute (2.9) into (2.10), $I_n(t)$ can be rewritten as

$$\begin{aligned} I_n(t) &= \sum_{i=0}^{n-1} 2^{ki} \{2^k S_{2^{i+2}\lambda_0}(f, k-1, t) - S_{2^{i+1}\lambda_0}(f, k-1, t)\} \\ &\quad - \sum_{i=0}^{n-1} 2^{ki} \{2^k S_{2^{i+1}\lambda_0}(f, k-1, t) - S_{2^i\lambda_0}(f, k-1, t)\} \\ &= 2^{kn} \{S_{2^{n+1}\lambda_0}(f, k-1, t) - S_{2^n\lambda_0}(f, k-1, t)\} \\ &\quad - 2^k \{S_{2\lambda_0}(f, k-1, t) - S_{\lambda_0}(f, k-1, t)\}. \end{aligned} \tag{2.12}$$

Hence

$$\begin{aligned} &(2^n \lambda_0)^k \| S_{2^{n+1}\lambda_0}(f, k-1, \cdot) - S_{2^n\lambda_0}(f, k-1, \cdot) \|_{C[a,b]} \\ &\quad \leqslant \lambda_0^k \| I_n(t) \|_{C[a,b]} + M' \leqslant M_2. \end{aligned}$$

To prove our saturation result, we need two more lemmas.

LEMMA 2.7. *Let*

$$B_m(\lambda, u) = \int_0^\infty W(\lambda, t, u)\, t^m\, dt. \tag{2.13}$$

Then (1) *$B_m(\lambda, t)$ is a polynomial in u and $1/\lambda$;* (2) *the degree of $B_m(\lambda, t)$ in u is m;* (3) *the coefficient of u^m in $B_m(\lambda, u)$ is* 1.

Proof. This lemma can be proved by using Lemma 2.2 and changing the variables. However, the following direct argument may be clearer.

First, we have

$$
\begin{aligned}
B_m(\lambda, u) &= 1 && m = 0,\\
&= u + (2/\lambda) && m = 1,\\
&= u^2 + (6/\lambda)\, u + (6/\lambda^2) && m = 2.
\end{aligned}
$$

That is, the assertion holds for $m = 0$, 1, and 2. Proceeding to the induction proceudre, first, we integrate $B_m(\lambda, u)$ by parts twice to obtain

$$
\begin{aligned}
B_m(\lambda, u) &= \int_0^\infty W(\lambda, t, u)\, t^m \, dt\\
&= -\frac{1}{m+1} \int_0^\infty (DW(\lambda, t, u))\, t^{m+1}\, dt\\
&= \frac{1}{(m+1)(m+2)} \int_0^\infty (D^2 W(\lambda, t, u)\, t^{m+2}\, dt.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\int_0^\infty [\mathbb{P} W(\lambda, t, u)]\, t^m \, dt - \int_0^\infty W(\lambda, t, u)\, t^{m+1}\, dt\\
&\quad = \int_0^\infty \left[\left(\frac{D^2}{\lambda^2} + \frac{2}{\lambda} D\right) W(\lambda, t, u)\, t^{m+1}\, dt\right.\\
&\quad = \frac{m(m+1)}{\lambda^2} B_{m-1}(\lambda, u) - \frac{2(m+1)}{\lambda} B_m(\lambda, u).
\end{aligned}
$$

Since $\mathbb{P} W(\lambda, t, u) = W(\lambda, t, u)\, u$, the above relation implies that

$$
B_{m+1}(\lambda, u) = u B_m(\lambda, u) + \frac{2(m+1)}{\lambda} B_m(\lambda, u) + \frac{m(m+1)}{\lambda^2} B_{m-1}(\lambda, u).
$$

The lemma follows readily by induction.

LEMMA 2.8. *Let* $0 < a < b < \infty$. *If* $f \in C_N[0, \infty)$, *and* $g \in C_0^\infty$, *with* supp $g \subset (a, b)$, *then*

$$
|\lambda^{k+1} \langle [S_{2\lambda}(f, k, t) - S_\lambda(f, k, t)], g(t)\rangle| \leq K \|f\|_{C_N}, \tag{2.14}
$$

where K is a constant depending only on g and its derivatives.

Proof. The proof is standard. First we observe

$$\begin{aligned}\langle S_\lambda(f, t), g(t)\rangle &= \int_0^\infty \int_0^\infty \{W(\lambda, t, u) f(u) g(t)\} \, du \, dt \\ &= \int_{\operatorname{supp} g} \int_0^\infty \{\cdots\} \, du \, dt \\ &= \int_{\operatorname{supp} g} \int_a^b \{\cdots\} \, du \, dt + \|f\|_{C_N} \cdot o(\lambda^{-(k+1)}) \\ &= \int_0^\infty \int_a^b \{\cdots\} \, du \, dt + \|f\|_{C_N} \cdot o(\lambda^{-(k+1)}),\end{aligned}$$

by the corollary to Lemma 2.2. Now, by Fubini's theorem, we change the order of integration, and then expand $g(t)$ by Taylor's formula to obtain

$$\begin{aligned}\langle S_\lambda(f, t), g(t)\rangle &= \sum_{\gamma=0}^{2k+2} \frac{1}{\gamma!} \int_a^b \int_0^\infty W(\lambda, t, u) f(u) g^{(\gamma)}(u)(t-u)^\gamma \, dt \, du \\ &\quad + \int_a^b \int_0^\infty W(\lambda, t, u) f(u) \epsilon(t, u)(t-u)^{2k+2} \, dt \, du \\ &\quad + o(\lambda^{-(k+1)}) \|f\|_{C_N} \\ &= I(\lambda) + J(\lambda) + o(\lambda^{-(k+1)}) \|f\|_{C_N}.\end{aligned}$$

Observing that $|\epsilon(t, u)| = (2/(2k+2)!) |g^{(2k+2)}(\xi)| \leqslant (2/(2k+2)) \|g^{(2k+2)}\|_\infty$, and $|f(u)| \leqslant e^{Nb} \|f\|_{C_N}$, it follows that

$$\begin{aligned}|J(\lambda)| &\leqslant M \|f\|_{C_N} \int_0^\infty \int_a^b W(\lambda, t, u)(t-u)^{2k+2} \, du \, dt \\ &\leqslant M \|f\|_{C_N} \left\{\int_0^{b+1} \int_a^b + \int_{b+1}^\infty \int_a^b\right\} = (J_1 + J_2) M \|f\|_{C_N}.\end{aligned}$$

Further, $J_1 \leqslant M_1 \lambda^{-(k+1)}$ by Lemma 2.2, since t is bounded; and, again by the same lemma,

$$\begin{aligned}J_2 &\leqslant \int_{b+1}^\infty \int_a^b W(\lambda, t, u)(t-u)^{2k+2} \left(\frac{t}{t} \cdot \frac{u}{b}\right)^{2m} du \, dt \\ &\leqslant M_2 \lambda^{-(k+1)} \int_{b+1}^\infty \frac{t^{k+m+1}}{(t-b)^{2m}} \, dt.\end{aligned}$$

By choosing $m > k+3$, we find that $J_2 \leqslant M_2' \lambda^{-(k+1)}$.

On the other hand,

$$I(\lambda) = \int_a^b \int_0^\infty W(\lambda, t, u) \sum_{\gamma=0}^{2k+2} \Psi_\gamma(u)\, t^\gamma \, dt\, du,$$

where

$$\Psi_\gamma(u) = \sum_{m=\gamma}^{2k+2} (-1)^{m-\gamma} \frac{1}{\gamma!(m-\gamma)!} f(u)\, g^{(m)}(u)\, u^{m-\gamma}.$$

Now by Lemma 2.7 and relation (2.8), we have

$$\sum_{j=0}^{k+1} \alpha(j, k)\, I(2^j\lambda) = O(\lambda^{-(k+1)}).$$

Therefore, combining the above estimates, we have,

$$\begin{aligned} &|\lambda^{k+1}\langle[S_{2\lambda}(f, k, t) - S_\lambda(f, k, t)], g(t)\rangle| \\ &\quad = \left| \lambda^{k+1} \cdot \sum_{j=1}^{2k+2} \alpha(j, k)\langle S_{2^j\lambda}(f, t), g(t)\rangle \right| \\ &\quad = O(1)\, \|f\|_{C_N}. \end{aligned}$$

Our main result in saturation is the following theorem.

THEOREM 2.8. *Let* $f \in C_N[0, \infty)$, *and let* $0 < a < a_1 < b_1 < b < \infty$. *Denote* $I(f, \lambda, k, a, b) = \lambda^{k+1} \| S_\lambda(f, k, t) - f(t)\|_{C[a,b]}$. *Then the following implications* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *and* (4) $\Rightarrow$ (5) $\Rightarrow$ (6) *hold.*

(1) $I(f, \lambda, k, a, b) = O(1)$; (2) $f^{(2k+1)} \in A.C.\ (a, b)$ *and* $f^{(2k+2)} \in L_\infty[a, b]$; (3) $I(f, \lambda, k, a, b) = O(1)$; (4) $I(f, \lambda, k, a, b) = o(1)$; (5) $f \in C^{2k+2}(a, b)$ *and* $\sum_{j=k+1}^{2k+2} Q(j, k, t)\, f^{(j)}(t) = 0$ *in* (a, b) *where* $Q(j, k, t)$ *are polynomials in* t; (6) $I(f, \lambda, k, a_1, b_1) = o(1)$.

Here, all $O(1)$ *and* $o(1)$ *terms are with respect to* λ *when* $\lambda \to +\infty$.

Proof. The method used here goes back to DeLeeuw and Lorentz (see, e.g., [6] or [7]).

Let $J(\lambda, f, k, t) = \lambda^{k+1}[S_{2\lambda}(f, k, t) - S_\lambda(f, k, t)]$. Clearly, condition (1) implies

$$\|J(\lambda, f, k, t)\|_{C[a,b]} \leqslant M. \tag{1*}$$

In the following we prove that (1*) implies (2) by induction on k.

This proposition is true for $k = 0$ by assumption. Assume that it is true for $k - 1$, then by Lemma 2.6 we have $f^{(2k)} \in L_\infty[a, b]$ as an intermediate result.

Next, condition (1*) shows that $\{J(\lambda, f, k, t)\}_\lambda$ is bounded in $C[a, b] \subset L_\infty[a, b]$. Since $L_\infty[a, b]$ is the dueal of $L_1[a, b]$, $\{J(\lambda, f, k, t)\}_\lambda$ is weak* compact. That is, there are $h \in L_\infty[a, b]$ and subnet $\{\lambda_i\}$ of $\{\lambda\}$ such that $\{J(\lambda_i, f, k, t)\}$ converges to h in the weak* topology. In particular, for any $g \in C_0$, with supp $g \subset (a, b)$, we have

$$\langle J(\lambda_i, f, k, t), g(t)\rangle \to \langle h, g\rangle . \tag{2.15}$$

On the other hand, for any $\phi \in C^{2k+2}[a, b]$, Lemma 2.5 implies

$$\begin{aligned}\langle J(\lambda_i, \phi, k, t), g(t)\rangle &\to \langle P_{2k+2}(D)\, \phi, g\rangle \\ &= \langle \phi, P^*_{2k+2}(D)\, g\rangle .\end{aligned}$$

where $P^*_{2k+2}(D)$ is the adjoint of $P_{2k+2}(D)$ (in this case, it is simply a result of integration by parts).

Since $C^{2k+2}[a, b] \cap C_N[0, \infty)$ is dense in $C_N[0, \infty)$, there exists $\{\phi_p\}$ in $C^{2k+2}[a, b] \cap C_N[0, \infty)$ converging to f in $\|\cdot\|_{C_n}$-Norm. Now, using Lemma 2.8, we have

$$\lim_{p\to\infty} \lim_{\lambda_i\to\infty} \langle J(\lambda_i, \phi_p, k, t), g(t)\rangle = \lim_{\lambda_i\to\infty} \langle J(\lambda_i, f, k, t), g(t)\rangle . \tag{2.17}$$

Combining (2.15), (2.16), and (2.7), we get $\langle h(t), g(t)\rangle = \langle f(t), P^*_{2k+2}(D)\, g(t)\rangle$ for all $g \in C_0^\infty$. This implies $P_{2k+2}(D) f(t) = h(t)$ since they are equal as generalized functions. Recall that $f^{(2k)} \in L_\infty[a, b]$, from the above differential equation we obtain $f^{(2k+2)} \in L_\infty[a, b]$.

This Proves (1) ⇒ (2). The proof for the implication (4) ⇒ (5) is similar and the other implications are calculational. We shall omit the rest of the proof.

3. A Direct Theorem

A direct theorem can be easily deduced from the properties proved in the last section. We estimate the degree of approximation of $S_\lambda(f, k, t)$ to $f(t)$ in terms of the general moduli of smoothness of f which is defined by

$$\omega_k(f; h, a, b) = \sup\{|\Delta_t^k f(x)| ; |t| \leq h, x + kt \in [a, b]\}, \tag{3.1}$$

where

$$\Delta_h^k f(x) = \sum_{\gamma=0}^{k} (-1)^{k-\gamma} \binom{k}{\gamma} f(x + \gamma h). \tag{3.2}$$

THEOREM 3.1. *Let $f \in C_N[0, \infty)$ and $0 < a < a_1 < b_1 < b < +\infty$. Then there exists a constant M such that*

$$\| S_\lambda(f, k, t) - f(t)\|_{C[a_1,b_1]} \leqslant M(\omega_{2k+2}(f; \lambda^{-(1/2)}, a, b) + \lambda^{-(k+1)} \| f \|_{C_N}). \quad (3.3)$$

Proof. Consider a function $g \in C^{2k+2}$ defined by

$$g(x) = \frac{1}{\binom{2k+2}{k+1} \eta^{2k+2}} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \Big[(-1)^k \bar{\Delta}^{2k+2}_{u_1+\cdots+u_{2k+2}} f(x) + \binom{2k+2}{k+1} f(x)\Big] du_1 \cdots du_{2k+2}, \quad (3.4)$$

where $(k+1)\eta < \min(a_1 - a, b - b_1)$, $\eta = \lambda^{-(1/2)}$, and

$$\bar{\Delta}_h^{2n} = \sum_{j=0}^{2n} (-1)^\gamma \binom{2n}{\gamma} f(x + (n-j)h).$$

First observe that, from the definition of g, we have

$$| f(x) - g(x) | \leqslant \frac{1}{\binom{2k+2}{k+1} \eta^{2k+2}} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} | \bar{\Delta}^{2k+2}_{u_1+\cdots+u_{2k+2}} | du_1 \cdots du_{2k+2}$$
$$\leqslant M\omega_{2k+2}(f; \eta(k+1), a, b) \leqslant (k+1) M\omega_{2k+2}(f; \eta, a, b)$$
$$= M_1\omega_{2k+2}(f; \lambda^{-(1/2)}, a, b), \quad (3.5)$$

for all $x \in [a_1, b_1]$.

On the other hand, we claim that

$$\| g \|_{C[a_1,b_1]} \leqslant M \| f \|_{C[a,b]} \leqslant M' \| f \|_{C_N}. \quad (3.6)$$

and

$$\| g^{(2k+2)} \|_{C[a_1,b_1]} \leqslant M\lambda^{(k+1)}\omega_{2k+2}(f; \lambda^{-(1/2)}, a, b). \quad (3.7)$$

Inequality (3.6) is clear. In order to show (3.7), notice that

$$(-1)^k \binom{2k+2}{k+1} \lambda^{-(k+1)} g(x)$$
$$= \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \Big\{\bar{\Delta}^{2k+2}_{u_1+\cdots+u_{2k+2}} f(x) + (-1)^k \binom{2k+2}{k+1} f(x)\Big\} du_1 \cdots du_{2k+2}$$
$$= \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \Big[\sum_{i=0}^{2k+2} (-1)^i \binom{2k+2}{i} f\Big\{x + (k+1-i) \sum_{j=1}^{2k+2} u_j\Big\}$$
$$+ (-1)^k \binom{2k+2}{k+1} f(x)\Big] du_1 \cdots du_{2k+2}. \quad (3.8)$$

Further, observe that

$$\frac{d^{2k+2}}{dx^{2k+2}} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left[f\left(x + \sum_{i=1}^{2k+2} u_i\right) + f\left(x - \sum_{i=1}^{2k+2} u_i\right) \right] du_1 \cdots du_{2k+2}$$
$$= 2\bar{\Delta}_\eta^{2k+2} f(x), \tag{3.9}$$

and

$$\omega_{2k+2}(f, |k+1-i| \eta) \leq |k+1-i| \omega_{2k+2}(f, \eta) = M_i \omega_{2k+2}(f, \lambda^{-(1/2)}). \tag{3.10}$$

Using (3.8), (3.9), and (3.10)

$$\| g^{(2k+2)} \|_{C[a_1,b_1]} = \frac{\eta^{-(2k+2)}}{\binom{2k+2}{k+1}} \left\| \sum_{i=0}^{k} (-1)^i \binom{2k+2}{i} \cdot 2\bar{\Delta}^{2k+2}_{(k+1-i)\eta} f(x) \right\|_{C[a_1,b_1]}$$
$$\leq \eta^{-(2k+2)} \frac{2}{\binom{2k+2}{k+1}} \sum_{i=0}^{k} \binom{2k+2}{i} (k+1-i)$$
$$\leq \omega_{2k+2}(f; \eta, a, b),$$

which proves (3.7).

Now we prove (3.3). By linearity of $S_\lambda(\cdot, k, t)$, we get

$$S_\lambda(f, k, t) - f(t)$$
$$= S_\lambda(f - g, k, t) - (g(t) - f(t)) + (S_\lambda(g, k, t) - g(t))$$
$$= I_1(t) + I_2(t) + I_3(t).$$

It follows from (3.5) that

$$\| I_2(t) \|_{C[a_1,b_1]} \leq M\omega_{2k+2}(f; \eta, a, b) = M\omega_{2k+2}(f; \lambda^{-(1/2)}, a, b). \tag{3.12}$$

The estimation for $I_1(t)$ follows from the corollary to Theorem 2.2: indeed, we have,

$$I_1(t) \leq \sum_{j=0}^{k} |C(j,k)| \int_0^\infty W(2^j\lambda, t, u) \, |f(u) - g(u)| \, du,$$

and

$$\int_0^\infty W(\lambda, t, u) \, |f(u) - g(t)| \, du$$
$$= \int_{|u-t| \leq \delta} + \int_{|u-t| > \delta}$$
$$\leq \| f - g \|_{C[a,b]} + K_m \lambda^{-m} \| f \|_{C_N}.$$

where $\delta < \min(a_1 - a, b - b_1)$.

Hence, using (3.5),

$$\| I_1(t)\|_{C[a_1,b_1]} \leqslant M_2\omega_{2k+2}(f; \lambda^{-(1/2)}, a, b) + K_m\lambda^{-m} \|f\|_{C_N}. \qquad (3.13)$$

It remains to estimate $I_3(t)$. By Taylor's formula, we write

$$g(u) = \sum_{i=0}^{2k+1} \frac{g^{(i)}(t)}{i!}(u-t)^i + \frac{1}{(2k+2)!} g^{(2k+2)}(\xi),$$

where ξ between u and t. We separate the integral into two parts as in estimating $I_1(t)$, then use Lemma 2.2 and relation (2.8), we obtain

$$\| S_\lambda(g, k, t) - g(t)\|_{C[a_1,b_1]} \leqslant M\lambda^{-(k+1)} \sum_{i=k+1}^{2k+2} \| g^{(i)} \|_{C[a_1,b_1]} + K_m\lambda^{-m} \| g \|_{C_N}.$$

Since $\| g^{(i)} \|_{C[a_1,b_1]} \leqslant M(\| g \|_{C[a_1,b_1]} + \| g^{(2k+2)} \|_{C[a_1,b_1]})$, and choosing $m \geqslant k+1$, we have further that

$$\| S_\lambda(g, k, t) - g(t)\|_{C[a_1,b_1]} \leqslant M'\lambda^{-(k+1)}(\| g^{(2k+2)} \|_{C[a,b]} + \| g \|_{C_N}).$$

Using (3.6), (3.7), and the defining relation (3.4) of g, the estimate of I_3 is

$$\begin{aligned}\| I_3(t)\|_{C[a_1,b_1]} = \| S_\lambda(g, k, t) - g(t)\|_{C[a_1,b_1]} &\leqslant M(\omega_{2k+2}(f; \lambda^{-(1/2)}, a, b) \\ &\quad + \lambda^{-(k+1)} \|f\|_{C_N}). \end{aligned} \qquad (3.14)$$

Combining (3.12), (3.13), and (3.14), we obtain (3.3).

4. The Inverse Theorems

Let $0 < \alpha < 2$, and let Lip*(α; $C[a, b]$) be the Zygmund class of functions

$$\text{Lip}^*(\alpha; C[0, 1]) = \{f \in C[0, 1]; \omega_2(f; h, a, b) \leqslant Mh^\alpha\}, \qquad (4.1)$$

where $\omega_2(f; h, a, b)$ is the second modulus of smoothness of f defined in (3.1).

Our inverse theorem is for $S_\lambda(f, 0, t) = S_\lambda(f, t)$. Analogous theorems for general k's can be proved similarly.

THEOREM 4.1. *Let* $0 < a_i < a_{i+1} < b_{i+1} < b_i < +\infty$, $i = 1, 2$, $0 < \alpha < 2$, *and suppose* $f \in C_N[0, \infty)$. *Then in the following statements, the implications* (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) *hold.*

(1) $\| S_\lambda(f, t) - f(t)\|_{C[a_1,b_1]} = O(\lambda^{-\alpha/2})$;

(2) $f \in \text{Lip}^*(\alpha; C[a_2, b_2])$;

(3) $f \in \mathrm{Lip}(\alpha; C[a_2, b_2])$, *if* $0 < \alpha < 1$, $f \in \mathrm{Lip}^*(1; C[a_2, b_2])$, *if* $\alpha = 1$, $f' \in C^1[a_2, b_2]$ *and* $f' \in \mathrm{Lip}(\alpha - 1; C[a_2, b_2])$, if $1 < \alpha < 2$;

(4) $\| S_\lambda(f, t) - f(t)\|_{C[a_3, b_3]} = O(\lambda^{-\alpha/2})$.

The equivalence of (2) and (3) is known (cf., [9, pp. 257, 333, and 337]). The implication of (3) to (4) follows from the direct theorem proven in the last section.

There are two major steps in proving that (1) implies (2).

(I) We first reduce the original problem to the following one, a special case when f has compact support inside some interior interval $[a', b']$ of (a_1, b_1).

THEOREM 4.1′. *Let* $0 < a < b < +\infty$, $0 < \alpha < 2$. *If* $f \in C_0$ *with* supp $f \subset (a, b)$, *then the following statements are equivalent*

(1′) $\| S_\lambda(f, t) - f(t)\|_{C[a, b]} = O(\lambda^{-\alpha/2})$,

(2′) $f \in \mathrm{Lip}^*(\alpha; C[a, b])$.

The proof of this reduced version will be given in the next step. In this step we show that Theorem 4.1′ implies Theorem 4.1.

Let a', a'', b', and b'' be chosen so that $a_1 < a' < a'' < a_2$ and $b_2 < b'' < b' < b_1$. Also, let $g \in C_0^\infty$ be such that supp $g \subset [a'', b'']$ and $g(x) = 1$ on $[a_2, b_2]$. In order to prove the assertion, it suffices to show, assuming Theorem 4.1′ is true, that the condition $\| S_\lambda(f, t) - f(t)\|_{C[a_1, b_1]} = O(\lambda^{-\alpha/2})$ would imply $\| S_\lambda(fg, t) - fg(t)\|_{C[a', b']} = O(\lambda^{-\alpha/2})$. The proof of the last assertion is also divided into two steps.

(I.1) First assume $0 < \alpha \leq 1$. For $t \in [a', b']$, we have

$$\begin{aligned} S_\lambda(fg, t) &- f(t)\, g(t) \\ &= g(t)[S_\lambda(f, t) - f(t)] + \int_{a_1}^{b_1} W(\lambda, t, u)\, f(u)[\, g(u) - g(t)]\, du + o(1/\lambda) \\ &= I_1(t) + I_2(t) + o(1/\lambda), \end{aligned} \tag{4.2}$$

where the $o(1/\lambda)$ term is uniform for $t \in [a', b']$ by the corollary to Lemma 2.2.

The assumption $\| S_\lambda(f, t) - f(t)\|_{C[a_1, b_1]} = O(\lambda^{-\alpha/2})$ yields the estimate

$$\| I_1(t)\|_{C[a', b']} \leqslant \| g \|_\infty \cdot \| S_\lambda(f, t) - f(t)\|_{C[a', b']} \leqslant M_1 \lambda^{-\alpha/2}. \tag{4.3}$$

The estimate of $I_2(t)$ follows by the mean value theorem

$$I_2(t) = \int_{a_1}^{b_1} W(\lambda, t, u)\, f(u)[\, g'(\xi)(u - t)]\, du. \tag{4.4}$$

Hence, by Lemma 2.2 and Cauchy–Schwarz inequality,

$$\| I_2 \|_{C[a',b']} = O(\lambda^{-1/2}) \leqslant O(\lambda^{-\alpha/2}). \tag{4.5}$$

Combining the above estimates, we conclude

$$\| S_\lambda(fg, t) - fg(t) \|_{C[a',b']} = O(\lambda^{-\alpha/2}).$$

Therefore, Theorem 4.1 holds for at least $0 < \alpha \leqslant 1$.

(I.2) Now assume $1 < \alpha < 2$. In this case we choose two more points x and y such that $a_1 < x < a'$ and $b' < y < b_1$. Let $\delta \in (0, 1)$. We shall prove the assertion for $1 < \alpha < 2 - \delta$. Since δ is arbitrary, we may then conclude the assertion holds for all $\alpha < 2$.

First notice that, by the result of (I.1), the condition $\| S_\lambda(f, t) - f(t) \|_{C[a_1,b_1]} = O(\lambda^{-\alpha/2})$ implies $f' \in \mathrm{Lip}[1 - \delta, C[x, y])$. Now for $t \in [a', b']$, we form

$$\begin{aligned} S_\lambda(fg, t) &- f(t)\, g(t) \\ &= g(t)[S_\lambda(f, t) - f(t)] + f(t)[S_\lambda(g, t) - g(t)] \\ &\quad + \int_x^y W(\lambda, t, u)[f(u) - f(t)][g(u) - g(t)]\, du + o(\lambda^{-1}) \\ &= J_1(t) + J_2(t) + J_3(t) + o(\lambda^{-1}), \end{aligned} \tag{4.6}$$

where the $o(\lambda^{-1})$ term is uniform for $t \in [a', b']$ (corollary to Lemma 2.2).

The facts that $\| J_1 \|_{C[a',b']} = O(\lambda^{-\alpha/2})$ follows from the assumption, and $\| J_2 \|_{C[a',b']} = O(\lambda^{-1}) \leqslant O(\lambda^{-\alpha/2})$ by Lemma 2.4. Also, since $| f(u) - f(t) | \leqslant M \, | u - t |^{1-\delta}$ and $g(u) - g(t) = g'(\xi)(u - t)$, using Jensen's inequality and Lemma 2.2, we obtain $\| J_3 \|_{C[a',b']} = O(\lambda^{-(2-\delta)/2}) \leqslant O(\lambda^{-\alpha/2})$.

(II) We prove Theorem 4.1′ in this step. Since $\operatorname{supp} f \subset (a, b)$, we may choose a', a'', b', b'' in such a way that $a < a' < a'' < b'' < b' < b$, $\operatorname{supp} f \subset [a'', b'']$. Let $\mathscr{G}$ denote the class

$$\mathscr{G} = \{ g \in C_0^2, \operatorname{supp} g \subset [a', b'] \} \tag{4.7}$$

and define the Peetre K-function by

$$K(\xi, f) = \inf\{\| f - g \| + \xi \, \| g'' \| \,;\, g \in \mathscr{G}\}, \tag{4.8}$$

where $0 < \xi \leqslant 1$.

In the following we shall only prove conditions (1′) implies (2′) in Theorem 4.1′.

LEMMA 4.2. *Condition* (1′) *of Theorem* 4.1 *implies*

$$K(\xi, f) \leqslant M_0(\lambda^{-\alpha/2} + \lambda \xi K(\lambda^{-1}, f)). \tag{4.9}$$

Proof. Since supp $f \subset [a'', b'']$, there is an $h \in \mathscr{G}$, such that

$$\left\| h^{(i)}(t) - \frac{d^i}{dt^i} S_\lambda(f, t) \right\|_{C[a,b]} \leqslant M'\lambda^{-1}.$$

$i = 0$ and 2. Therefore

$$K(\xi, f) \leqslant 3M'\lambda^{-1} + \| f(t) - S_\lambda(f, t)\|_{C[a,b]} + \xi \| S''_\lambda(f, t)\|_{C[a,b]} .$$

Hence, it is sufficient to show that there exists an M, such that, for each $g \in \mathscr{G}$

$$\| S''_\lambda(f, t)\|_{C[a,b]} \leqslant M\lambda\{\| f - g \| + \lambda^{-1} \| g'' \|\}. \tag{4.10}$$

But, by linearity of $(d^2/dt^2)\, S_\lambda(\cdot, t)$,

$$\| S''_\lambda(f, t)\|_{C[a,b]} \leqslant \| S''_\lambda(f - g, t)\|_{C[a,b]} + \| S''_\lambda(g, t)\|_{C[a,b]} . \tag{4.11}$$

Moreover, differentiating the kernel $W(\lambda, t, u)$ directly gives

$$\frac{\partial^2}{\partial t} W(\lambda, t, u) = e^{-\lambda(t+u)} \sum_{n=1}^{\infty} \frac{\lambda^{2n}u^{n-1}t^{n-2}}{n!\,(n-1)!}\{(n - \lambda t)^2 - n\} + \lambda^2 e^{-\lambda t}\, \delta(u). \tag{4.12}$$

Now, since

$$\begin{aligned}
&\int_0^\infty \left| \frac{\partial^2}{\partial t^2} W(\lambda, t, u) \right| du \\
&\quad \leqslant \int_0^\infty e^{-\lambda(t+u)} \sum_{n=1}^{\infty} \frac{\lambda^{2n}u^{n-1}t^{n-2}}{n!\,(n-1)!} [(n - \lambda t)^2 + n]\, du + \lambda^2 e^{-\lambda t} \\
&\quad = \frac{\lambda^2}{t^2} \int_0^\infty W(\lambda, t, u)(u - t)^2\, du = \frac{2\lambda}{t},
\end{aligned}$$

we have

$$\| S''_\lambda(f - g, t)\|_{C[a,b]} \leqslant (2\lambda/a) \| f - g \| = M_1\lambda \| f - g \|. \tag{4.13}$$

On the other hand, by Lemma 2.2, the degree of $S_\lambda(x^m, t)$ is m, it follows That $S''_\lambda(1, t) = S''_\lambda(x, t) = 0$. Therefore,

$$\begin{aligned}
S''_\lambda(g, t) &= \int_0^\infty \left[\frac{\partial^2}{\partial t^2} W(\lambda, t, u)\right] g(u)\, du \\
&= \int_0^\infty [\cdots]\, g(t) + g'(t)(u - t) + g''(\xi)(u - t)^2\, du \\
&= \int_0^\infty [\cdots]\, g''(\xi)(u - t)^2\, du
\end{aligned}$$

and

$$\begin{aligned}\|S_\lambda''(g,t)\|_{C[a,b]} &\leq \|g''\| \cdot \left\| \int_0^\infty \left| \frac{\partial^2}{\partial t^2} W(\lambda, t, u) \right| (u-t)^2\, du \right\|_{C[a,b]} \\ &\leq \|g''\| \cdot \left\| \int_0^\infty e^{-\lambda(t+u)} \sum_{n=1}^\infty \frac{\lambda^{2n} u^{n-1} t^{n-2}}{n!\,(n-1)!} \right. \\ &\quad \times \left. [(n-\lambda t)^2 + n](u-t)^2\, du + (\lambda t)^2 e^{-\lambda t} \right\|_{C[a,b]} \\ &\leq \|g''\| \cdot \left\| e^{-\lambda t} \sum_{n=0}^\infty \frac{(\lambda t)^{n-2}}{n!} [(n-\lambda t)^2 + n]^2 \right\|_{C[a,b]} \\ &\leq \|g''\| \cdot \left\| 6 + \frac{4}{\lambda t} \right\|_{C[a,b]} \leq M_2 \|g''\|.\end{aligned} \tag{4.14}$$

Combining (4.13) and (4.14), we obtain (4.10). This completes the proof of the lemma.

LEMMA 4.3. *Relation* (4.9) *implies*

$$K(\xi, f) \leq M\xi^{\alpha/2} \qquad \textit{for some constant } M > 0. \tag{4.15}$$

The proof of this lemma is standard and can be found in [1].

Therefore, to complete the proof of Theorem 4.1′ (and hence the proof of Theorem 4.1) it is sufficient to prove the following lemma.

LEMMA 4.4. *Relation* (4.15) implies

$$f \in \mathrm{Lip}^*(\alpha; C[a, b]). \tag{4.16}$$

Proof. Let $0 < |\delta| \leq h$. Then for any $g \in \mathscr{G}$, we have

$$\begin{aligned}|\Delta_\delta^2 f(t)| &\leq |\Delta_\delta^2(f(t) - g(t))| + |\Delta_\delta^2 g(t)| \\ &\leq 4\|f-g\| + \delta^2 \|g''\|.\end{aligned}$$

Thus

$$\omega_2(f; h, a, b) \leq 4K(h^2, f) \leq 4Mh^\alpha, \tag{4.17}$$

or $f \in \mathrm{Lip}^*(\alpha; C[a, b])$.

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